Journal of Applied Fluid Mechanics, Vol. 8, No. 3, pp. 391-398, 2015.
Available online at www.jafmonline.net, ISSN 1735-3572, EISSN 1735-3645.

A Flow Model for the Settling Velocities of Non Spherical Particles in Creeping Motion. Part III. Slender Bodies, the Stream Functions, the Flow and the Momentum Equation

Y. Mendez

Independent Research Group, Ottawa, Ontario, K2B 5W9

Email: info@independentresearchgroup.ca

(Received November 09, 2012; accepted June 09, 2014)

ABSTRACT

This paper follows previous work regarding the settling velocity of non spherical particles in creeping motion. In this paper, we summarize the flow model, present solutions for the slender plate and the cylinder (Stoke's paradox), demonstrate the application for euhedral pseudo hexagonal plates (KGa-1) and show the match to the experimental data. In addition, we derive the stream function for the sphere, the slender cylinder and the plate, develop the relationships to compute the flow about a settling particle, back calculate the momentum equation and examine the result

Keywords: Settling; Wall shear; Expansion; Non spherical particles; Stokes' paradox; Pressure gradient; Stream function; Creeping flow.

NOMENCLATURE

a	edge length of an hexagon
ASS	specific surface area
b	½ the thickness of a slender plate
e	tributary ratio
e_{max}	maximum tributary ratio
e_p	pseudo tributary ratio
e_{pmax}	maximum pseudo tributary ratio
G_s	specific gravity of solid
G_f	specific gravity of fluid
h	tributary volume
h_{max}	maximum tributary volume
H	thickness of an hexagonal prism
L	length of a cylinder
g	acceleration due to gravity
p	pressure gradient
P_f	potential pressure gradient
Q_s	flow about the equator of an spherical particle
r	radius of a sphere
r_c	radius of a cylinder
r_s	radius of a solid sphere
r_{sc}	radius of a solid cylinder
r_{sp}	radius of a pseudo solid sphere
R	radius of a spherical expansion
R_p	radius of a spherical pseudo expansion
R_{max}	maximum radius of a spherical expansion
R_{pmax}	maximum radius of a pseudo expansion
T	tributary volume about any particle
T_{ws}	submerged tributary weight
u	velocity
$v(a,b)$	volume as a function of a and b
V_s	settling velocity of a sphere
V_{nsp}	settling velocity of a non-spherical particle
V_{sc}	settling velocity of a slender cylinder
V_{sp}	settling velocity of a slender plate
φ	mass expansion rate per unit velocity gradient
ρ_f	density of the fluid
ρ_s	density of the solids
μ	viscosity
ξ	spherical expansion
ξ_{max}	maximum spherical expansion
ξ_c	cylindrical expansion
ξ_{pmax}	maximum spherical pseudo expansion
τ	shear stress
τ_w	wall shear
τ_{wc}	wall shear of a slender cylinder
ψ_c	stream function for a cylindrical expansion
ψ_p	stream function for a planar expansion
ψ_s	stream function for a spherical expansion

1. INTRODUCTION

This paper follows previous work, Mendez, Y. (2011 and 2013), in the derivation of a flow model for the settling velocity of non-spherical particles in creeping motion, which in further discussion will be referred to as the Wall Shear-Pressure Gradient-Expansion (WPE) model. The model has been evaluated with published experimental data in water and other fluids with kinematic viscosities in the range of 1×10^{-6} and 6×10^{-7} m^2/s.

Interesting insights reached in its derivation allow for a rational solution for difficult geometries. A long sought after goal. The introduction of the Wall Shear τ_w, the Ambient Expansion ξ and the Pressure Gradient P_f resulting from the mobilization of shear stress across the expansion yield the computational devices to apply a simple rational to address the said difficult geometries.

In this paper we present solutions for two slender bodies, namely, the flat plate and the cylinder and other selected geometries. We also solve for hexagonal prisms and derive the relationships for the stream functions for these geometries and the flow about the equator of a sphere.

Remarkably, the solutions include Stokes' paradox. A paradox resulting in criticism to Stokes mathematical formulation and proposed solutions by Oseen. C. W. (1910). The assumptions behind the WPE differ from the original formulation by Stokes and Oseen and other subsequent discussions. The fundamental difference resides in the assumption that the dynamics of the problem are controlled by shear stress and that the fundamental solution is tailored to answer the question of what is the magnitude of the pressure gradient in the free fluid that challenge the shear stress of the fluid? and what are the shape, mass and volume relationships of the ambient expansion with respect to the surface, volume and mass of the solid?. And those fundamental differences are what the author can add to the discussion.

Our intent is to summarize the model, demonstrate its application and provide additional insights as an aid to its application for further research or to solve science or engineering problems.

The limits of the applicability have not been established. With regard to the fluid properties, good agreement to the experimental data has been found in fluids with low viscosities (water, cyclohexane and toluene). The creeping regime is expected to be highly sensitive of the adhesive and viscous forces relative to the density of the fluid or kinematic viscosity, as such; good agreement is expected for fluids with low kinematic viscosities. For all data sets the Reynolds number is less than 1. According to the experimental data the break down of the relationships appear to occur at a velocity gradient in the order of $160\ s^{-1}$ in water at $20\ C^o$. As a reference, for specific gravity of solids $Gs = 2.65$ this velocity gradient develops on a sphere of $30\ \mu m$ radius.

2. THE WALL SHEAR

The WPE has been derived by first characterizing the driving force. It has been envisioned that the submerged weight is to be transferred to the fluid by shear stress in the form of a tributary mass of solids per square meter of particle which is easily accomplished by dividing the submerged weight of solids by the area of the particle. The inverse of the Specific Surface Area *ASS* of the particle can be seen to be kg/m^2 which can be converted into submerged tributary weight T_{ws} over the square meter of particle (N/m^2) or wall shear τ_w made available at the surface of the particle as a result of gravity. The quantity sought after. For example a sphere of radius r_s and density ρ_s of known volume $4/3\pi r^3$and area $4\pi r^2$ can be seen to have $ASS=(r/3)\rho_s$ and submerged tributary weight over the square meter of its surface $T_{ws} = (r/3)(\rho_s-\rho_f)g = \tau_w$ in N/m^2. Where ρ_f is the density of the fluid. We thus can evaluate in comparable grounds the driving force for any geometry. For example a slender plate of thickness t and a slender cylinder of diameter d_c, both of the same density must satisfy the equality of their tributary volumes $Tvp = t/2$ and $Tvc = d_c/4$ respectively to mobilize the same driving τ_w. The cylinder diameter needs to be twice the thickness of the plate. The diameter of a sphere needs to be three times the thickness of the plate. It will be shown later that these two slender bodies and the sphere settle at different velocities. Although the tributary weight is not enough to decide on the velocity and do not define the geometry exactly it is very restrictive of the magnitude of the physical thickness of the particles. Formulas for the volume and surface area for a wide variety of geometries, including polyhedra, have been derived and from the formulas, the tributary weight is of easy access so that there is no additional contribution on presenting more examples.

3. THE EXPANSION

For spheres, the physical dimensions and the volume and area relationships defined by two concentric spheres is not difficult to understand. Our goal is to envision the volume and area relationships in the per square meter basis. If we trim a *30* degrees cone out of the two concentric spheres the tributary mass of solids, the spherical expansion of fluid between the spheres and the difference between the physical dimensions and the volume and area relationships are rather apparent. This is the physical environment portrayed by the computations associated with the WPE with one additional ingredient: the size of the spherical expansion needs to be such to accomplish the computation of the shear stress at the wall through the dynamics of viscosity along a spherical tributary volume. This volumetric relationship is a key for applying the dynamics of viscosity. Failure to establish the mathematical relationships associated with the free expansion as they relate to the geometry of the settling object in per square meter basis will result in failure in application of viscosity to solve the problem, as in previous attempts. For spheres this is accomplished as follows:

$$\mu \frac{du}{dr} = \frac{-\tau_w}{h_{\max}} \frac{\left(r^3 - R^3\right)}{3r^2} \qquad (1)$$

where:

h_{max} = the maximum tributary volume

μ = viscosity[

u = velocity

r = the radius of any spherical surface within the ambient fluid,

R = the radius of the spherical system containing the ambient fluid and the solid sphere. Where the velocity is zero.

Note the ratio of the shear stress to its tributary volume. The computation provides the same value of pressure gradient P_f along the entire ambient fluid. It is denoted as h_{max} for simplicity but is also spherical in geometry with radius R which is unknown at this point. The pressure gradient is hence unknown too. It can be seen that we can easily work our way to the momentum equation as will be done later on this paper. Upon integration and the known boundary condition we reach:

$$u = \frac{-\tau_w}{2\mu h_{\max}}\left(\frac{r^2}{3} + \frac{2R^3}{3r} - R^2\right) \tag{2}$$

$$\text{as } h_{\max} = \frac{\left(R^3 - r_s^3\right)}{3r_s^2} \tag{3}$$

Eq. 2 turns into

$$V_s = \frac{-\tau_w}{2\mu}\frac{3r_s^2}{\left(R^3 - r_s^3\right)}\left(\frac{r_s^2}{3} + \frac{2R^3}{3r_s} - R^2\right) \tag{4}$$

for the computation of the settling velocity Vs at the wall of the sphere. It can be seen that measured settling velocities of known spherical geometry can provide evidence of the size of the ambient expansion and the value of the pressure gradient. The experimental data lead to a value of *991Pa/m* for fluid properties, density and viscosity at *15* C^o ($\mu = 0.001139$ *Pa-s* and $\rho_f = 999.3$ *kg/m*^{3}). Note that the boundary condition portrayed by Eq. 4 computes the velocity at the wall but the entire velocity profile can be computed from Eq. 2 which can be written as:

$$u = \frac{-P_f}{2\mu}\left(\frac{r^2}{3} + \frac{2R^3}{3r} - R^2\right) \tag{5}$$

Equation 4 can be written in a more friendly manner by defining the ratio of the volume of the spherical ambient expansion to the volume of the solid sphere e_{max} (tributary volume) as:

$$e_{\max} = \frac{\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r_s^3}{\frac{4}{3}\pi r_s^3} = \frac{R^3 - r_s^3}{r_s^3} \tag{6}$$

For

$$R = r_s\left(1 + e_{\max}\right)^{\frac{1}{3}} \tag{7}$$

Substituting our findings in Eq. 4 we obtain

$$V_s = \frac{-P_f r_s^2}{2\mu}\left(1 + \frac{2e_{\max}}{3} - \left(1 + e_{\max}\right)^{\frac{2}{3}}\right) \tag{8}$$

We have denoted the factor between the brackets as the maximum spherical expansion ξ_{max}. It is very informative. It tells us that the velocity at the wall from the velocity profile portrayed by Eq. 5 can be computed for any particle size using a single multiplier to the fraction shown in Eq. 8 for the given fluid properties. This is how Stokes' Law works so well for different particle sizes. For the fluid properties above and specific gravity of solids $Gs = 2.65$ $e_{max} = 16.34$, i.e. the volume of the ambient expansion is 16.34 times the volume of a settling particle of *2,65* specific gravity.

Some implications were not apparent upon derivation of the relationships. The pressure gradient does not vary with the size or variation of the specific gravity of two identical settling particles as it depends only in the fluid properties. Differences in size or specific gravity only change the size of the ambient expansion. This is an implication of equilibrium. Since P_f times tributary volume equal shear stress, P_f times the volume of fluid in the ambient expansion equal the submerged weight of the particle. This can be seen when we note:

$$\frac{\tau_w}{h_{\max}} = P_f \tag{9}$$

For spheres,

$$h_{\max} = \frac{\frac{4}{3}\pi R_{\max}^3 - \frac{4}{3}\pi r_s^3}{4\pi r_s^2} = \frac{R_{\max}^3 - r_s^3}{3r_s^2} \tag{10}$$

Which turns out to be,

$$h_{\max} = \frac{r_s^3\left(1 + e_{\max}\right) - r_s^3}{3r_s^2} = \frac{e_{\max} r_s}{3} \tag{11}$$

As such we can write,

$$P_f = \frac{\frac{r\rho_s}{3}\left(\rho_s - \rho_f\right)g}{\frac{e_{\max} r}{3}} = \frac{\left(\rho_s - \rho_f\right)g}{e_{\max}} \tag{12}$$

The expansion will be the same for a given specific gravity of solids. It will change in size in 16.34 times the volume of the sphere or tributary volume but the expansion remains the same. It can also be seen that it will change in size for different specific gravity of solids. In our discussion below it will also be seen that the pressure gradient and hence the expansion, change with the variation in temperature as it depends on the density, viscosity and the mass expansion of the fluid about the velocity gradient.

4. THE PRESSURE GRADIENT

The velocity gradient is a mass expansion law as seen under the implications of the momentum equation:

$$\mu \frac{du}{dr} = P_f h(r) \Rightarrow \frac{\mu}{P_f} = \frac{h(r)}{du/dr} \quad (13)$$

Where *h(r)* is the tributary volume as a function of a physical dimension *r*, normal to the surface driving the shear stress. The tributary volume in a fluid of density ρ_f is also a tributary mass M_t in kg/m^2 as follows

$$M_t = h\rho_f \quad (14)$$

The mass expansion law can thus be written as

$$\frac{\mu\rho_f}{P_f} = \frac{M_t}{du/dr} \quad (15)$$

For verification we bring forward the results the experimental values for sand particles *0.5* to *25.0* µm (*Gs* taken as *2.65*) from Zegzhda (1934), Arkangel'skii (1935), and Sarkisyan (1958) reproduced by Cheng (1997). These results were also presented in Mendez (2011) and presented here for ease of reference to demonstrate how the derivation is accomplished. Additional experimental results and validation can be found in Mendez (2013).

Table 1 Calculated settling velocities from Eq. 13 and experimental values for sand in water at *15 C*° from Zegzhda (1934), Arkangel'skii (1935) and Sarkisyan (1958) reproduced by Cheng, N. (1997). Average error less than 1%.

r (*µm*)	Computed from Eq. 12 τ_w*(mPa)*	Computed from Eq. 8 *Vs (µm/s)*	Measured *(µm/s)*
0.5	2.70	0.56	0.57
2.5	13.49	14.11	14.10
5.0	26.97	56.46	56.50
10.0	53.94	225.83	223.00
25.0	134.85	1411.42	1410.00

Based on these results we have determine that the tributary ratio $e_{max} = 16.34$. The tributary volume about the $r_s = 25\ \mu m$ particle is thus $e_{max}\, r_s/3 = 1.362 \times 10^{-4}\ m^3/m^2$ and the velocity gradient $\tau_w/\mu = 118.4\ s^{-1}$. The tributary mass about the velocity gradient of $118.4\ s^{-1}$ is then computed as $1.362 \times 10^{-4}\ m^3/m^2 \times \rho_f = 0.1361\ kg/m^2$ which then divided by the velocity gradient give us the tributary mass per unit velocity gradient $Mt/(du/dr)=\varphi = 1.149 \times 10^{-3}\ kg\text{-}s/m^2$. This value can then be verified to be satisfied for the rest of the particles i.e. The same φ can be obtained by dividing the tributary mass of fluid by the corresponding velocity gradient for each particle.

As one is able to compute the ratio on the right side of Eq. 15, *Mt/(du/dr)*, further denoted φ, which based on the evidence in the above paragraph, does not vary, is hence plausible to compute the pressure gradient from the fluid properties and φ. The variation in the settling velocity with temperature can then be determined, not only based on the variation of viscosity and the density of the fluid but also in the change of size of the expansion. Velocities computed based on this variation of the pressure gradient with temperature have been verified to be in better agreement with the experimental data than Stokes' law for variation in temperature from *15* to *20 C*° Mendez, Y. (2013). The computation of the pressure gradient is thus proposed as

$$\frac{\mu\rho_f}{\varphi} = P_f \quad (16)$$

for conditions and properties similar to those described in our validations.

5. SLENDER BODIES

The concepts of expansion and pressure gradient summarized above provide simple rational solutions to difficult geometries. On the other hand the wall shear also furnish a rational to describe the driving force for different geometries. For instance, we consider a *50 µm* micrometer diameter particle of *Gs = 2.65* in water at *20 C*° ($\mu = 0.001003$ and $\rho_f = 998.3\ kg/m^3$). This particle has been observed to settle at $Vs = 0.00166\ m/s$ within the results from Zegzhda (1934), Arkangel'skii (1935) and Sarkisyan (1958) reproduced by Cheng (1997). The wall shear $\tau_w = (r_s/3)(\rho_s\text{-}\rho_f)g = 0.135\ N/m^2$. The pressure gradient for the fluid properties is computed as $Pf = (\mu\rho f)/\varphi = 871\ N/m^2$ and the maximum tributary volume $e_{max} = (\rho_s\text{-}\rho_f)g/p_f = 18.6$. This section proposed solutions for a slender plate and a slender cylinder mobilizing the same driving force as this sphere within the same fluid and temperature.

In the creeping regime we know from the first integration of the momentum equation that we should be able to compute the shear stress from the tributary volume as

$$\mu \frac{du}{dr} = P_f h(r) \quad (17)$$

Where *r* is a physical dimension normal to the solid surface. Our first geometry is the slender plate. The volume of a slender plate of thickness *t* can be seen to have a volume *t* and a solid tributary volume $t/2 = b_s$ in m^3/m^2. The wall shear $\tau_w = (\rho_s\text{-}\rho_f)gb_s$ of $0.135\ N/m^2$ is mobilized by a plate of $b_s = 8.33\ \mu m$. The relationship to be satisfied is $r_s/3=b_s$, i.e. a slender plate is to be of thickness *1/3* the diameter of the sphere to mobilize the same wall shear for solids with the same specific gravity. The tributary ratio informs that the volume of the fluid expansion must be *18.6* times the tributary volume of the particle to mobilize the same shear stress. For the slender plate, the tributary volume about the flat particle is also flat and extends to a maximum distance $h_{max} = bs + bs(18.6)$ in meters from the center of the flat particle. Equation 17, for the slender plate is thus written as

$$\mu \frac{du}{dh} = P_f \left(\frac{h - h_{\max}}{1 \times 1} \right) \quad (18)$$

thus

$$u = \frac{-P_f}{2\mu}\left(h^2 - 2hh_{\max} + h_{\max}^2\right) \tag{19}$$

after solving for the boundary condition, $u = 0$ when $h = h_{max}$. The entire velocity profile is available from $h = bs$ to $h = h_{max}$ by means of Eq. 19. We can use the useful conclusions of the expansion: as $b_s(1+e_{max}) = h_{max}$, Eq. 14 solves for the velocity of the slender plate Vsp, for $h = b_s$ at the wall as

$$V_{sp} = \frac{-P_f b_s^2}{2\mu} e_{\max}^2 \tag{20}$$

We can solve for the velocity of all plates of a given specific gravity within water at 20 C^o using an appropriate multiplier to the fraction in Eq. 20.

For a slender cylinder of radius r_{sc} and length L, the tributary weight of solids per square meter of cylinder or wall shear τ_{wc} can be computed as

$$\tau_{wc} = \frac{\pi r_{sc}^2 L}{2\pi r_{sc} L}\left(\rho_s - \rho_f\right)g = \frac{r_{sc}}{2}\left(\rho_s - \rho_f\right)g \tag{21}$$

To satisfy for the same wall shear as our sphere $r_s/3 = r_{sc}/2$. $r_{sc} = 16.67\ \mu m$. The cylindrical tributary volume hc of radius Rc will take the form

$$h_c = \frac{\pi r_c^2 H - \pi R_c^2 H}{2\pi r_c H} = \frac{r_c^2 - R_c^2}{2r_c} \tag{22}$$

Equation 18 can thus be written for the slender cylinder as

$$\mu \frac{du}{dr_c} = P_f\left(\frac{r_c^2 - R_c^2}{2r_c}\right) \tag{23}$$

and solved as

$$u = \frac{-P_f}{2\mu} \times \left(\frac{r_c^2}{2} + R_c^2 \ln\left(\frac{R_c}{r_c}\right) - \frac{R_c^2}{2}\right) \tag{24}$$

for the boundary conditions inclusive.

The tributary ratio allow us to write

$$\pi R_c^2 L = \pi r_{sc}^2 L + \pi r_{sc}^2 L e_{\max} \Rightarrow R_c = r_{sc}\left(1 + e_{\max}\right)^{\frac{1}{2}} \tag{25}$$

The settling velocity of the slender cylinder Vsc as computed from the cylindrical expansion ξ_c takes the form

$$V_{sc} = \frac{-P_f r_{sc}^2}{4\mu} \times \left(\ln\left(1 + e_{\max}\right) + e_{\max} \ln\left(1 + e_{\max}\right) - e_{\max}\right) \tag{26}$$

Figure 1 present the velocity profile for the three expansions across identical tributary volumes (holding the same volume per square meter but different geometry). Although some implications from Fig. 1 will be apparent for the reader, further comments and insights about Fig. 1 will arise upon our discussion for other geometries.

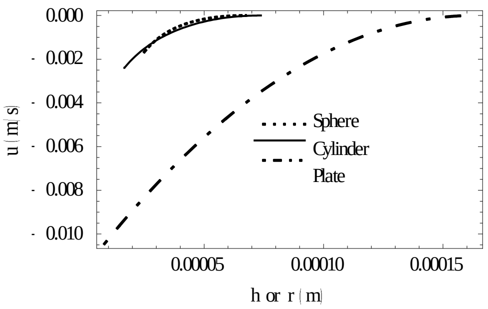

Fig. 1. Velocity profiles for three expansions having the same tributary volume but different geometry and mobilizing the same wall shear.

6. OTHER GEOMETRIES

According to the previous developments the WPE offers a simple rational solution to embrace difficult geometries from equilibrium considerations. We need to highlight the following for a given set of solid and fluid properties:

There is always a need to characterize the particle in terms of its tributary mass and wall shear accurately.

There is always a need to characterize the geometry of the expansion in per square meter basis as it relates to the geometry of the solid particle.

The computation of shear stress through the dynamics of viscosity is in equality with the computation of pressure gradient times tributary volume. Since, for different sizes the driving shear stress increases in proportion to the tributary volume of solids, there is a corresponding increase of the tributary volume of fluid. From the mathematical stand point this effect is accomplished by the tributary volume.

The maximum expansion devise assists on the computation of the velocity at the wall of the particle obeying the previous volume relationship but there is always a velocity profile involved

Figure 1 shows that it does not suffice to characterize the wall shear only without the geometry as the physical dimensions and the velocity are highly dependent on the geometry.

With the exceptions of spheres and the slender bodies in Fig. 1 characterization of the tributary volume on a single dimension is not a feasible goal as it is as ambitious as describing all geometries based on a single dimension. On the other hand, characterization of the tributary volume and the dynamics based on two dimensions is extremely challenging. Also, Fig. 1 informs that slender bodies are not realistic for a wide variety of naturally occurring materials whose specific gravities range between 2.5 and 2.9 as the expansions are large, rendering it unrealistic for even high aspects ratios. However, the requirements set out in the WPE are that the expansion is sufficiently large to mobilize the driving shear stress and that the geometry of the entire system correlates in the per square meter basis. These requirements move along general understanding of fluids. The statements of equilibrium velocities, equilibrium pressure gradient and even equilibrium geometry of the fluid are accomplished within the WPE. From the stand point of the WPE it is not difficult to satisfy these requirements when we can provide an appropriate geometry model for the particles. An equivalent sphere has been the goal of plenty of research. Somewhat reluctantly, the author has accepted this being the goal of this section because in essence the approach presented below to determine a pseudo sphere is fairly distinct from previous attempts. We will refer to the equivalent solid sphere as the pseudo sphere of radius r_{sp} concentric with the pseudo expansion of radius Rp. Simply put, our goal is to determine a sphere whose surface area is the same as the non spherical particle within an spherical expansion that mobilize the same shear stress as the non spherical particle. The volume of the non spherical particle times e_{max} computes the volume of the ambient expansion required to mobilize the submerged weight of the particle and the equality of area of the non spherical particle $Ansp$ with the area of the pseudo solid sphere give us the radius $Ans = 4\pi(rps)^2$. We denote for convenience the volume of any non spherical particle as $v(a,b)$ and $s(a,b)$ the area of the same particle to define the pseudo tributary volume e_{pmax} as:

$$e_{p\max} = \frac{e_{\max} v(a,b)}{\frac{4}{3}\pi r_{ps}^3} \tag{27}$$

which allow us to compute the velocity across a pseudo expansion ξ_{pmax} that is flatter and mobilize the same wall shear of the non spherical particle in the per square meter basis as it relates to the entire tributary volume. Flatter than the tributary volume about a sphere having the same specific gravity and wall shear of the non spherical particle.

The settling velocity of the pseudo sphere Vsp having the same shear stress and velocity $Vnsp$ as the non spherical particle is hence written as

$$V_{nsp} = \frac{-P_f r_{sp}^2}{2\mu} \times \left(1 + \frac{2e_{p\max}}{3} - \left(1 + e_{p\max}\right)^{\frac{2}{3}}\right) \tag{28}$$

Note that this is the end result from the fact that the volume of the pseudo sphere $4/3\pi(rsp)^3$ times $(1+e_{pmax}) = 4/3\pi(Rp)^3$ but the entire velocity profile is available as

$$u = \frac{-P_f}{2\mu}\left(\frac{r^2}{3} + \frac{2R_p^2}{3r} - R_p^2\right) \tag{29}$$

from $r = r_{sp}$ to $r = R_p$.

As an example, Georgia Kaoline KGa-1 has been widely characterized in the literature. A coarse particle of KGa-1 may be expected to be a pseudo hexagonal prism about $2\ \mu m$ in major dimension Żbik et al. (2007) and about $0.15\ \mu m$ thick. 2.65 for Gs is reasonable for the mineral. We will assume hexagonal prisms as the geometry model, whose edge length a is the same as the radius of a circle enclosing the prism. The edge length a of our particle is hence $1\ \mu m$ and its thickness $H = 0.15\ \mu m$. The surface area $S = 3\sqrt{3}a^2+6aH$ and the volume $v = (3/2)\sqrt{3}a^2H$. The definitions lead to $ASS = 5.9\ m^2/gr$ and the wall shear $\tau_w = (v/s)(\rho s\text{-}\rho f)g = 0.00103\ N/m^2$. The radius of the pseudo sphere having the same area as the prism is computed from the equality $4\pi(r_{sp})^2 = 3\sqrt{3}a^2+6aH$

$$r_{sp} = \left(\frac{3\sqrt{3}a^2 + 6aH}{4\pi}\right)^{\frac{1}{2}} = 6.965 \times 10^{-7}\ m \tag{30}$$

For water at $20\ C^o$, $\mu = 0.001003\ Pa\text{-}s$ and the density $\rho_f = 998.3\ kg/m^3$. The mass expansion per unit velocity gradient $\varphi = 0.001149\ (kg\text{-}s)/m^2$ (independent of temperature as far as this research has verified). The pressure gradient is reached as

$$\frac{\mu\rho_f}{\varphi} = 871.4\frac{Pa}{m}$$

and the tributary volume

$$e_{\max} = \frac{(\rho_s - \rho_f)g}{P_f} = 18.59 \tag{31}$$

We compute 5.1188 as the pseudo tributary volume from Eq. 27. This is the end outcome of depositing the water held within the limits of the expansion about the prisms on the pseudo sphere and establishing the volume relationships for the pseudo sphere. The computed settling velocity $Vnsp$ for this particle is $0.225\ \mu m/s$ ($1.57\ \mu m$ equivalent diameter for reference). The colloidal fraction of KGa-1 is about $0.15\ \mu m$ thick and about $0.75\ \mu m$ in length (Żbik and Frost (2009)). For this particle (further denoted particle B), $a = 0.375\ \mu m$ and $ASS = 7.4\ m^2/gr$, which lead to $r_{sp} = 0.291\ \mu m$ and $Vnsp = 0.22\ \mu m/s$. Note that the pseudo tributary volume for this particle is 9.83, as it varies with the size of the particle for the same geometry (0.6 μm equivalent diameter for reference). In the following data sets, from Pruett and Webb (1993) and Żbik and Smart (1998), equivalent spherical diameters were computed from settling velocity experiments so that the settling velocity can be computed back (using temperature of 20C) from the reported equivalent diameter and compare. Pruett and Webb state that "SediGraph 5100 particle size measurements indicate KGa-IB is 57.8% <2 μm and 32.0% <0.5 μm, whereas KGa-1 is 47.3% <2 μm (Vnsp A=1.2 μm/s as compared with 3.6 μm/s back calculated for the 2 μm particle) and 21.4% <0.5 μm (Vnsp B=0.22 μm /s as compared with 0.22 μm /s back calculated for the 0.5 μm particle)". In addition note the consistent result with the following information: Żbik and Smart summarize the general description of KGa-1 as "Ninety percent by weight of the particles have an equivalent spherical diameter less than 2 *μm* with a median particle size of 0.7 *μm* (a particle having an aspect ratio of 8 and length 2a = 1.22 *μm* settles at the back calculated Vs of 0.44 *μm* /s for this particle) and specific surface area of 15.3 ± 0.5 m2/gr (BET nitrogen adsorption)". Pruett and Webb report BET surface area measurements of 8.4 and 11.7 m2/gr for KGa-1 and KGa-1B respectively. It can be seen that the agreement with the experimental data is remarkable.

We can thus write the relationship for all sizes within the geometry model satisfying for a given velocity as

$$r_{sp}^2 \xi_{p\max} = r_s^2 \xi_{\max} \tag{32}$$

where r_s is the radius of a solid sphere settling at the given velocity.

7. THE STREAM FUNCTION

For incompressible plane steady flows a stream function ψ exist. Setting $\psi = 0$ at the far end we obtain

$$\psi_s = \frac{-P_f}{2\mu}\left(\frac{2R^3}{3}\ln\left(\frac{R}{r}\right) + rR^2 - \frac{8R^3 + r^3}{9}\right) \tag{33}$$

$$\psi_c = \frac{-P_f}{2\mu}\left(\frac{2R_c^3}{3} - r_c R_c^2\left(\frac{1}{2} + \ln\left(\frac{R}{r}\right)\right) + \frac{r_c^3}{6}\right) \tag{34}$$

$$\psi_p = \frac{-P_f}{2\mu}\left(\frac{h_{\max}^3}{3} - \frac{h^3}{3} + h^2 h_{\max} - h h_{\max}^2\right) \tag{35}$$

for the sphere, the cylinder and the plate respectively.

8. THE FLOW

For all the geometries presented above, we can compute the flow across the tributary volume. This is the flow per meter. For example the flow about the equator of the sphere goes as

$$Q_s = \int_{r_s}^{R} \frac{-P_f}{2\mu}\left(\frac{r^2}{3} + \frac{2R^2}{3r} - R^2\right) dr \times (2\pi r_s) \Rightarrow Q_s = \left(\Big|_{r_s}^{R} \frac{-P_f}{2\mu}\left(\frac{r^3}{9} + \frac{2R^3\ln(r)}{3} - rR^2\right)\right) \times (2\pi r_s) \Rightarrow Q_s = \frac{-P_f}{2\mu}\left(\frac{2R^3}{3}\ln\left(\frac{R}{r_s}\right) + r_s R^2 - \frac{8R^3}{9} - \frac{r_s^3}{9}\right) \times (2\pi r_s) \tag{36}$$

which can also be written as a function of the tributary volume.

9. THE MOMENTUM EQUATION

We can work our relationships back to the momentum equation as:

$$\mu \frac{d^2u}{dh^2} = P_f \tag{37}$$

$$\mu \frac{d^2u}{dr_c^2} = P_f\left(\frac{r_c^2 + R_c^2}{2r_c^2}\right) \tag{38}$$

$$\mu \frac{d^2u}{dr^2} = P_f\left(\frac{r^3 + 2R^3}{3r^3}\right) \tag{39}$$

for the plate, the cylinder and the sphere respectively and examine our results. For greater insight we present the entire context for spheres

$$\mu\nabla^2 V - \nabla p - \rho g = 0 \tag{40}$$

A system that is three dimensional and spherical is expected to balance to an equilibrium condition such that

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 w}{\partial z^2} = \frac{d^2u}{dr^2} \tag{41}$$

It may not be apparent that in the quiescent fluid the second term is zero and the shear stress is opposed by the mass of the fluid only in the form of a pressure gradient of magnitude $\mathbf{I_f}\rho = P_f$ where the fluid is weightless in itself and gravity does not influence the motion from means other than the imbalance caused by the solid sphere. We can thus define P_f as the potential pressure gradient equal in all three directions. As such the third term should be written as $\rho\nabla I_f$ or ∇P_f. For P_f equal in all three directions the end result is

$$3\mu \frac{d^2u}{dr^2} = 3P_f\left(\frac{r^3 + 2R^3}{3r^3}\right) \tag{42}$$

The final aspect is to explain the reason why the product of the viscosity times the second rate of change of the velocity with respect to the physical dimension for the sphere and the cylinder does not yield a constant value of pressure. The answer is not difficult. It is simply because the rate of change of the tributary volume ∇T with respect to the physical dimension is not constant. As opposed to the flat expansion for which the constant value is satisfied. Implicit in these findings is the fact that there is some missing information in the momentum equation. Modification of the basic equations of fluid mechanics have been proposed previously by research scientists. The end result of these developments is that the momentum equation for incompressible viscous flows across free expansions within a quiescent continuum should be written as

$$\mu\nabla^2 V - P_f \nabla T = 0 \tag{43}$$

yielding Eq. 42 for spheres after the mathematical manipulation. Note that these are the dynamics within an expansion whose extents and shape allows for the mobilization of a driving shear stress by the fluid. This is where the pseudo sphere is distinct from previous attempts to develop an equivalent sphere. What we accomplish in this approach is to model the expansion, not the solid particle as it can be verified that that the pseudo sphere does not have the same density as the solid non spherical particle. In the essence we alter the rate of change of the tributary volume with respect to the physical dimension to match the flatter expansion about the non spherical particle. Such manipulation yield great flexibility to use Eq. 39 to solve for difficult geometries demonstrated in this paper.

10. Conclusion

The WPE model here presented offers great flexibility to solve difficult problems in low Reynolds number hydrodynamics. The model does not only reach a simple rational to solve for the settling velocity of non spherical particles, but also provides important insights regarding the geometric relationships associated with the mobilization of stress and force within fluid expansions. These findings also suggest that the strictly momentum terms of the momentum equation may be effected by similar terms.

Acknowledgments

To my father Luis Paulino Mendez 1925-2012 whose devotion to his children is shown in this paper. This research has been conducted entirely with the resources and time of the author. Through Independent Research Group the author is seeking financial support for many years of research projects already envisioned. Find us at www.independentresearchgroup.ca.

References

Arkangel’skii, B. V. (1935). Experimental Study of Accuracy of Hydraulic Coarseness scale of Particles. *Isv. NHG* 15, Moscow, Russia (in Russian).

Brady, P., R. Cygan and K. Nagy (1996). Molecular Controls on Kaolinite Surface Charge. Journal of Colloid Interface Science 183(2), 356-364.

Cheng, N. (1997). Simplified Settling Velocity Formula for Sediment Particle. *Journal of Hydraulic Engineering* 123(2), 149-152.

Chipera, S. and D. Bish (2001). Baseline Studies of the Clay Minerals Society Source Clays: Powder X-Ray Diffraction Analyses. Clays and Clay Minerals 49(5), 398-409.

Mendez, Y. (2011). A flow Model for the Settling Velocity of non Spherical Particles in Creeping Motion. *Journal of Applied Fluid Mechanics* 4(4), 65-75.

Mendez, Y. (2012). A flow Model for the Settling Velocity of non Spherical Particles in Creeping Motion. Part II. *Journal of Applied Fluid Mechanics* 5(4), 123-129.

Oseen, C. W. (1910). "Über die Stokes'sche formel, und über eine verwandte Aufgabe in der Hydrodynamik", Arkiv för matematik, astronomi och fysik vi (29).

Pruett, R. and H. Webb (1993). Sampling and analysis of KGa-1 B well-crystallized kaolin source clay. *Clays and Clay Minerals*. 41(4), 514-519.

Sarkisyan, A. A. (1958). Deposition of sediment in a turbulent stream. Izd. AN SSSR, Moscow, Russia (in Russian).

Żbik, M. and R. St. Smart (1998). Nanomorphology of kaolinites: comparative SEM and AFM studies. Clays and clay Minerals 46(2), 152-160.

Żbik, M., R. Frost, Y. Song, Y. Chen and J. Chen (2007), Transmission x-ray microscopy reveals the clay aggregate discrete structure in aqueous environment. Journal of Colloid Interface Science 319(2), 457-461.

Zegznda, A. P. (1934). Settlement of sand gravel particles in still water. fev. NIIG 12, Moscow, Russia (in Russian).